Popular Matchings and Optimality: Problem Sheet

In all the problems given below, $G = (A \cup B, E)$ is a bipartite graph where A is a set of agents and B is a set of jobs. Every vertex $u \in A \cup B$ has a strict ordering $\succ_u$ of its neighbors.

1. The problem is to decide if G has a stable matching that contains a given edge e. Show a polynomial time algorithm for this problem.

   [Update the Gale-Shapley algorithm to solve the above problem.]

   Suppose we are given k edges $e_1, \ldots, e_k$. Show a polynomial time algorithm to decide if G has a stable matching that contains all these k edges.

2. For any matching M in G, recall the edge weight function wt_M defined in the first lecture. For any edge ab, recall that $\mathsf{wt}_M(ab) \in \{0, \pm 2\}$ and $\mathsf{wt}_M(uu) \in \{0, -1\}$ for any $u \in A \cup B$.

   If M is popular then show there exists $\vec{\alpha} \in \{0, \pm 1\}^n$ (where $n = |A \cup B|$) such that the following three properties are satisfied.

   (a) $\alpha_a + \alpha_b \geq \mathsf{wt}_M(ab)$ for any edge ab;

   (b) $\alpha_u \geq \mathsf{wt}_M(uu)$ for any vertex u;

   (c) $\sum_{u \in A \cup B} \alpha_u = 0$.

3. For any popular matching M, we know there exists a vector $\vec{\alpha}$ as described in the above exercise. Let $U_0 = \{u \in A \cup B : \alpha_u = 0\}$ and let $U_1 = \{u \in A \cup B : \alpha_u \in \{\pm 1\}\}$. Define the following two subsets of M:

$$
\begin{aligned}
M_0 &= \{ab \in M : \alpha_a = \alpha_b = 0\}. \\
M_1 &= \{ab \in M : \alpha_a, \alpha_b \in \{\pm 1\}\}.
\end{aligned}
$$

   Show that $M = M_0 \cup M_1$. That is, for every edge $ab \in M$, show that the parities of α-values of a and b are the same.

4. We will prove that the following formulation describes the *stable matching polytope*, i.e., the convex hull of the edge incidence vectors of stable matchings. This proof is by Teo and Sethuraman [MOR, 1998].

   It is easy to check that the edge incidence vector of every stable matching in G satisfies constraints (1)-(3) where for any $u \in A \cup B$, $\mathsf{Nbr}(u)$ is the set of u's neighbors in G and $\delta(u) \subseteq E$ is the set of edges incident to u.

$$
\sum_{\substack{b' \in \mathsf{Nbr}(a) \\ b' : b' \succ_a b}} x_{ab'} + \sum_{\substack{a' \in \mathsf{Nbr}(b) \\ a' : a' \succ_b a}} x_{a'b} + x_{ab} \geq 1 \quad \forall ab \in E \tag{1}
$$

$$
\sum_{e \in \delta(u)} x_e \leq 1 \quad \forall u \in A \cup B \tag{2}
$$

$$
x_e \geq 0 \quad \forall e \in E. \tag{3}
$$

Consider the LP with objective function max $\sum_{e \in E} x_e$ subject to constraints (1)-(3). The following is the dual LP. The variables are y_e for $e \in E$ and z_u for $u \in A \cup B$.

$$\min \quad -\sum_{e \in E} y_e + \sum_{u \in A \cup B} z_u \quad \text{subject to}$$

$$- \sum_{\substack{b' \in \mathsf{Nbr}(a) \\ b' : b' \prec_a b}} y_{ab'} \; - \sum_{\substack{a' \in \mathsf{Nbr}(b) \\ a' : a' \prec_b a}} y_{a'b} - \; y_{ab} + z_a + z_b \;\; \geq \;\; 1 \quad \forall ab \in E$$

$$y_e \;\geq\; 0 \quad \forall e \in E \;\; \text{and} \;\; z_u \;\; \geq \;\; 0 \quad \forall u \in A \cup B.$$

Note that $v' \prec_u v$ denotes that v' is ranked worse than v in u's ranking of its neighbors.

Consider the following assignment of values to $\vec{y}$ and $\vec{z}$.

- Let $\vec{y} = \vec{x}$, where $\vec{x}$ is a feasible solution to the primal LP.
- For any $u \in A \cup B$: let $z_u = \sum_{e \in \delta(u)} x_e$.

(a) Show that this is a feasible solution to the dual LP.

(b) Use (a) to show that for any feasible solution $\vec{x}$ to the primal LP and any edge ab such that $x_{ab} > 0$, constraint (1) is tight for ab. That is:

$$\sum_{\substack{b' \in \mathsf{Nbr}(a) \\ b' : b' \succ_a b}} x_{ab'} + \sum_{\substack{a' \in \mathsf{Nbr}(b) \\ a' : a' \succ_b a}} x_{a'b} + \; x_{ab} \;\; = \;\; 1.$$

We are now ready to prove that every primal feasible solution $\vec{x}$ can be expressed as a convex combination of stable matchings in G. For any vertex u, let $\vec{x}_u$ be the vector corresponding to u's assignment in $\vec{x}$. Thus $\vec{x}_u = (x_e)_{e \in \delta(u)}$. If $\sum_{e \in \delta(u)} x_e < 1$ then add $(1 - \sum_{e \in \delta(u)} x_e)$ also as a coordinate in $\vec{x}_u$: so with fractional value $1 - \sum_{e \in \delta(u)} x_e$, the vertex u is assigned null.

- For each $a \in A$, let $\vec{x}'_a$ be the vector obtained from $\vec{x}_a$ by pruning all 0-values and reordering the entries in $\vec{x}_u$ in *decreasing* order of a's preference.
- For each $b \in B$, let $\vec{x}'_b$ be the vector obtained from $\vec{x}_b$ by pruning all 0-values and reordering the entries in $\vec{x}_u$ in *increasing* order of b's preference.

Form the table T whose rows are the reordered arrays $\vec{x}'_u$, for $u \in A \cup B$. Note that the table T has width 1. For any $t \in [0, 1)$, define the set M_t as follows:

- Draw the line ℓ_t at distance t from the left end of T.
- The line ℓ_t intersects (or touches the left boundary of) some cell in $\vec{x}'_u$ for each vertex u: call this cell $c_u(t)$.

$$\text{Let } M_t = \{e : e \in c_u(t) \text{ where } u \in A \cup B\}.$$

(d) Prove that M_t is a matching in G.

(e) Prove that M_t is a *stable* matching in G.

This shows that $\vec{x}$ is a convex combination of stable matchings. Thus constraints (1)-(3) describe the stable matching polytope of G.

5. The primal and dual LPs in the above problem are closely related. The LP considered in this exercise is even better—show that (LP1) is its own dual.

   In any feasible solution $\vec{x}$, every vertex u is completely matched. This is by assuming it is matched along the self-loop uu with fractional value $1 - \sum_{e \in \delta(u)} x_e$ where $\delta(u)$ is the set of edges incident to u (not counting the self-loop). Each vertex considers itself to be its last choice neighbor.

$$\text{minimize} \sum_{u \in A \cup B} \alpha_u \tag{LP1}$$

subject to

$$\alpha_a + \alpha_b \geq \left(\sum_{\substack{b' \in \mathsf{Nbr}(a) \cup \{a\} \\ b' : b' \prec_a b}} x_{ab'} - \sum_{\substack{b' \in \mathsf{Nbr}(a) \\ b' : b' \succ_a b}} x_{ab'} \right) + \left(\sum_{\substack{a' \in \mathsf{Nbr}(b) \cup \{b\} \\ a' : a' \prec_b a}} x_{a'b} - \sum_{\substack{a' \in \mathsf{Nbr}(b) \\ a' : a' \succ_b a}} x_{a'b} \right) \quad \forall\, ab \in E$$

$$\alpha_u \geq -\sum_{e \in \delta(u)} x_e \quad \forall\, u \in A \cup B$$

$$\sum_{e \in \delta(u) \cup \{uu\}} x_e = 1 \quad \forall u \in A \cup B \qquad \text{and} \qquad x_e \geq 0 \quad \forall e \in E \cup \{uu : u \in A \cup B\}.$$