Introduction to Popular Matchings: Problem Sheet

In all the problems given below, $G = (A \cup B, E)$ is a bipartite graph where A is a set of agents and B is a set of jobs. Every vertex in $A \cup B$ has a strict ordering or ranking of its neighbors.

1. Recall that a matching M in G is stable if there is no edge $ab \notin M$ such that the following conditions hold.

   - a is unmatched or prefers b to its partner in M and
   - b is unmatched or prefers a to its partner in M.

   Recall the Gale-Shapley algorithm: *agents propose and jobs dispose.* In more detail:

   1. Initialize $M = \emptyset$.
   2. while there exists at least one unmatched agent who has not been rejected by all its neighbors do:
      (a) let a be such an agent;
      (b) let b be a's topmost neighbor who has not yet rejected a.
      (c) a proposes to b.
      (d) If b is unmatched or prefers a to its partner in M then
         – if b is matched in M then it rejects its partner;
         – b accepts a's proposal.
      (e) Else b rejects a's proposal.
   3. Return M.

   Prove that M is a stable matching in G. In other words, prove the correctness of the Gale-Shapley algorithm.

2. Prove that every stable matching in G matches the same subset of vertices. That is, if vertex v is matched in one stable matching in G then v is matched in every stable matching in G. Thus all stable matchings in G have the same size.

3. Show that every stable matching in G is a min-size popular matching.

4. Prove that every max-size popular matching in G matches the same subset of vertices.

5. We saw that a stable matching M' in the red/blue graph G' corresponds to a max-size popular matching in G. That is, ignoring the edge colors in M', the resulting matching M is a max-size popular matching.

   Suppose we make *three* copies of every edge in G, i.e., every $e \in E$ has three copies e (red), e (blue), and e (green) in the new graph G''. The vertex set of G'' is $A \cup B$.

   - Every $a \in A$ prefers any red edge to any blue edge and any blue edge to any green edge.

- Every $b \in B$ prefers any green edge to any blue edge and any blue edge to any red edge.
- Within any color class, the preference order of any vertex is the original preference order in G.

Let M'' be a stable matching in G''. Let M be the matching in G obtained by ignoring edge colors in M''.

(a) Prove that $|M| \geq \frac{3}{4} \cdot |M_{\max}|$ where $M_{\max}$ is a maximum matching in G.

(b) Show that the matching M is not *very unpopular*, i.e., in a head-to-head election against any matching N, the number of votes for $N \leq 2 \cdot$ (the number of votes for M).

Research problems

1. Call a matching M *quasi-popular* if it satisfies the condition given in Exercise 5. That is, in a head-to-head election against any matching N, we have:

   $$\text{The number of votes for } N \leq 2 \cdot (\text{the number of votes for } M).$$

   Exercise 5 tells us that there is a linear time algorithm to find a quasi-popular matching of size $\geq \frac{3}{4} \cdot |M_{\max}|$ where $M_{\max}$ is a maximum matching in G.

   (a) Is this always a *max-size* quasi-popular matching?

   (b) If not, is there a polynomial time algorithm to find a max-size quasi-popular matching?

2. Is every popular matching in G also stable? Or does there exist a popular matching in G that is *not* stable? Since we know how to find a max-size popular matching and a min-size popular matching in linear time, if all popular matchings in G are not of the same size, then we can answer "no" in linear time.

   Suppose all popular matchings in G have the same size. Then the above algorithm is of no use. Can the above problem be solved in linear time even when all popular matchings have the same size?

   (There is an $O(m^2)$ algorithm known for this problem where $m = |E|$. So this problem can be solved in polynomial time.)