Popular Matchings and Optimality

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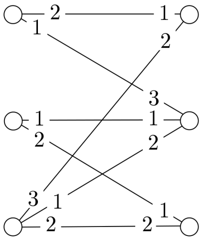
May 14, 2024

The input

A bipartite graph where every vertex has a strict ranking of its neighbors.



We seek a matching that pairs up vertices as happily as possible.

How do we formalize such a pairing?

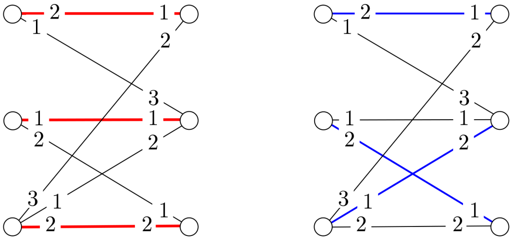
- Every agent a should be paired to the best possible job b such that

  b is willing to be matched to a.

Stability

The following property should hold for any agent/job v in our matching M:

- every neighbor ranked better than $M(v)$ is matched to a neighbor better than v.

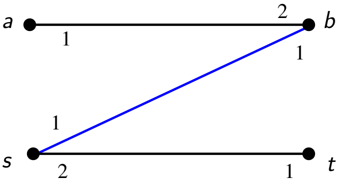


- The red matching satisfies this property; not the blue matching.
- The edge between the two middle vertices *blocks* the blue matching.
- A matching with no blocking edge is a stable matching.

Stable matchings

Stability is a very natural notion of "good matching".

The Gale-Shapley algorithm: agents propose and jobs dispose — this is a simple and clean algorithm.



As we saw, the above instance has a stable matching $\{sb\}$ of size 1.

- So the size of a stable matching might be only half the size of a maximum matching.
- This is the "price of stability".

Beyond stability

This motivates relaxing stability to a more *collective* or "democratic" notion.

Every vertex v has a ranking over all the possible matchings in G:

- $M \succ_v N$ if $M(v)$ is better than $N(v)$ in v's ranking;
- $M \prec_v N$ if $M(v)$ is worse than $N(v)$ in v's ranking;
- $M \sim_v N$ if $M(v) = N(v)$.

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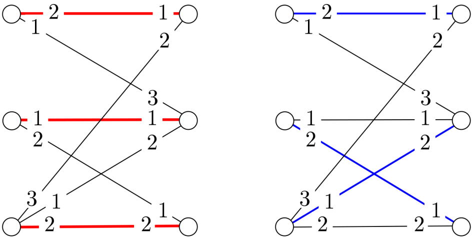
Any pair of matchings can be compared as follows:

- hold a head-to-head election between these two matchings.
- vertices are voters in this election.
- count the number of votes won by each matching.
- the matching with smaller number of votes loses this election.

The $\succeq$ and $\succ$ operators

Given two matchings M, N, we write $M \succeq N$ if

$$\underbrace{|\{v \in A \cup B : M(v) \succ_v N(v)\}|}_{\#\text{ of votes for } M} \geq \underbrace{|\{v \in A \cup B : N(v) \succ_v M(v)\}|}_{\#\text{ of votes for } N}$$



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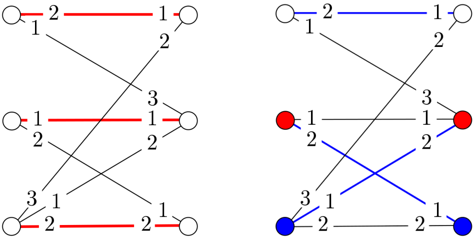

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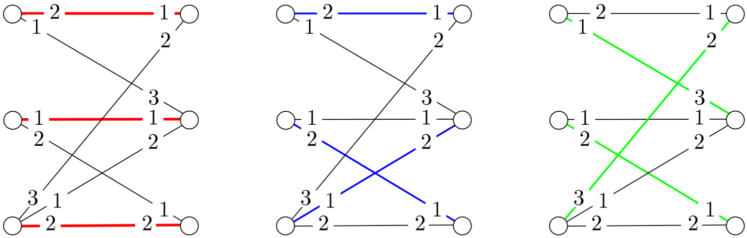

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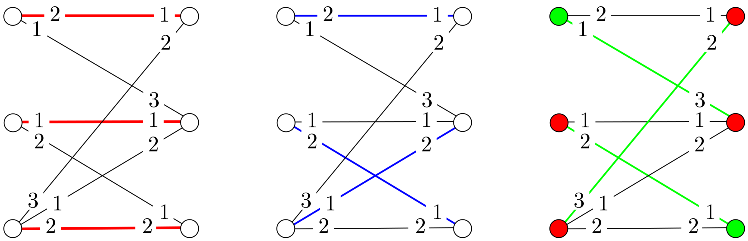

and $M \succ N$ if the inequality is strict (M defeats N).

The $\succeq$ and $\succ$ operators

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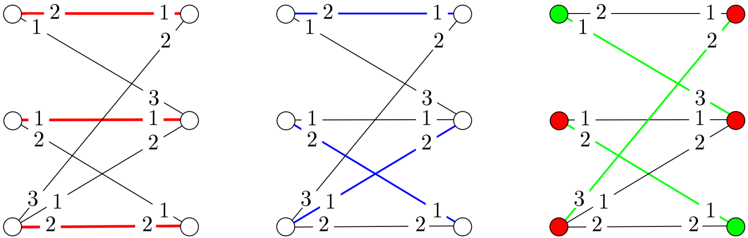


and $M \succ N$ if the inequality is strict (M defeats N).

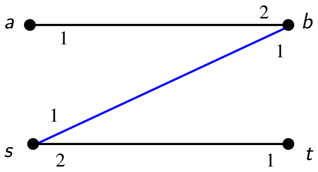
A matching M that does not lose any election is a popular matching.

Max-size popular matching

Every stable matching is popular [Gardenfors, 1975].

- Every stable matching is a min-size popular matching.

In the instance below $\{ab, st\}$ is a max-size popular matching.



Our goal now is to find the matching $\{ab, st\}$ here.

- We know that running the Gale-Shapley algorithm finds $\{sb\}$.

In order to find $\{ab, st\}$, a should get a second chance to propose to b.

- However b will again reject a.

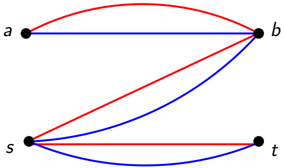
A new instance G'

IDEA. b should prefer a's second proposal to s's first proposal.

A new graph G' where every edge uv in G is replaced by uv and uv in G':

- that is, by two parallel edges: one red and the other blue.

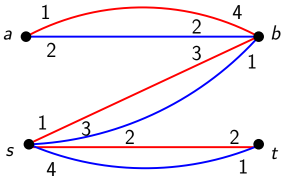
The corresponding graph G' is:



- Red edges correspond to *first-time* proposals.
- Blue edges correspond to *second-time* proposals.

A new instance G'

The graph G' with preferences is:



- Every agent prefers any red edge to any blue edge.
- Every job prefers any blue edge to any red edge.

The matching $\{ab, st\}$ is stable here.

- Ignoring colors, this is the desired matching $\{ab, st\}$.

Stable matchings in G'

Our algorithm in $G = (A \cup B, E)$

- Construct the red/blue graph $G' = (A \cup B, E')$.
- Run Gale-Shapley algorithm in G' to compute M'.
- Return the corresponding matching M in G.

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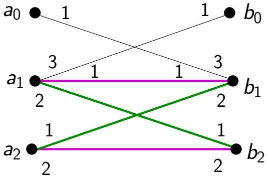
CLAIM. M is popular matching in G.

- Furthermore, M is more popular than any larger matching.

Thus M is a max-size popular matching in G.

Max-size popular matchings



There are two max-size popular matchings here: purple and green.

- Only the green matching occurs as a stable matching in the red/blue graph G'.
- The purple matching cannot be realized as a stable matching in G'.

Hence not every max-size popular matching occurs as a stable matching in G'.

Dominant matchings

Any stable matching in the red/blue graph G' has an interesting property:

- M is a popular matching that is more popular than all larger matchings.

DEF. Call a popular matching that defeats all larger matchings dominant.

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Dominant matchings in $G \equiv$ Stable matchings in the red/blue graph G'.
[Cseh and K, 2018]

- Given $e \in E$: is there a popular matching in $G = (A \cup B, E)$ with e?

Popularity with a forced edge

Forced edge e

1. Check if there is a stable matching in G with edge e.
2. Check if there is a dominant matching in G with edge e.
3. If the answer in steps 1 and 2 is no then return "no".

Popularity with a forced edge

FORCED EDGE e

1. Check if there is a stable matching in G with edge e.
2. Check if there is a dominant matching in G with edge e.
3. If the answer in steps 1 and 2 is no then return "no".

- Any popular matching decomposes into a stable part and a dominant part.
- This leads to the correctness of the above algorithm.

Popularity with a forced pair of edges

Given a pair of edges e_1, e_2 in G:

- Is there a popular matching in G with both e_1 and e_2?

There are polynomial time algorithms to determine if there is a stable matching with a given set $\{e_1, \ldots, e_k\}$ of edges [Gusfield and Irving, 1989].

- there need not be a stable / dominant matching with both e_1 and e_2.

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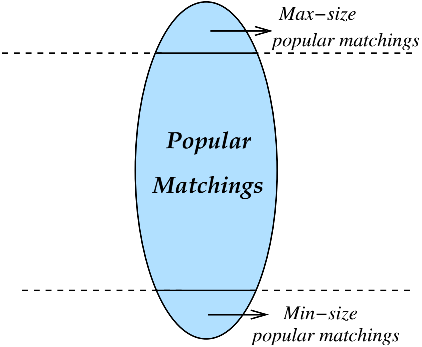
THEOREM. The above problem is NP-hard [Faenza, K, Powers, Zhang, 2019].

The easy subclasses

Finding a max-size (similarly, min-size) popular matching in G is easy.



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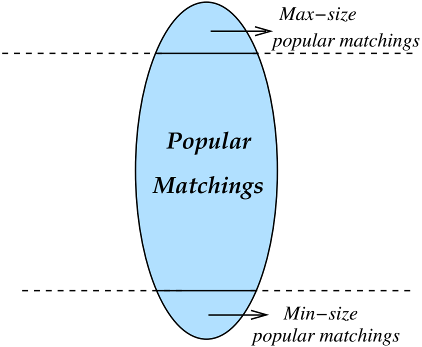

Algorithmic question: Is there *any* popular matching in G that is neither a max-size nor a min-size popular matching?

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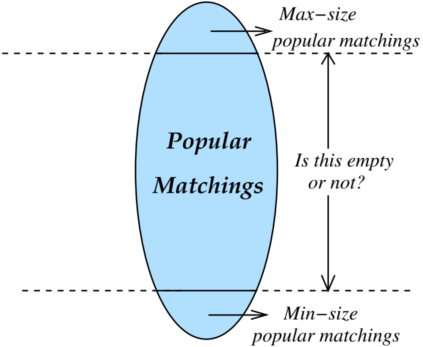

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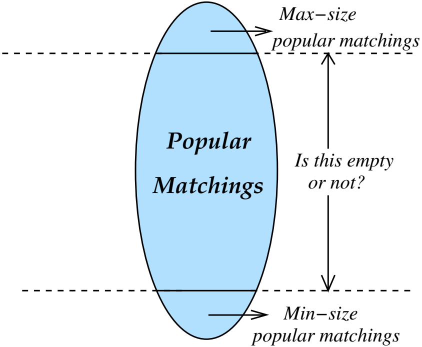

It is NP-hard to decide if G admits a popular matching that is neither a max-size nor a min-size popular matching.

Min-cost popular matchings

Suppose every edge has a cost.

- It is NP-hard to find a min-cost popular matching.

Finding a min-cost stable matching in G is easy.

- Efficient combinatorial [Irving, Leather, and Gusfield, 1987] and LP-based [Vande Vate, 1987; Rothblum, 1992] algorithms.
- The stable matching polytope has a linear-size description.

A relaxation of the min-cost popular matching problem:

- the min-cost popular mixed matching problem.

Mixed matchings

A mixed matching is a probability distribution over matchings, i.e.,

$$\Pi = \{(M_0, p_0), \ldots, (M_k, p_k)\},$$

where $M_0, \ldots, M_k$ are matchings in G and $\sum_i p_i = 1$ and $p_i \geq 0 \ \forall i$.

- A mixed matching is a lottery over matchings.

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- A mixed matching is a lottery over matchings.

For any two matchings M and N:

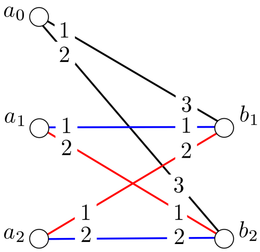
$$\text{let } \Delta(N, M) \;=\; \#\text{ of votes for } N - \#\text{ of votes for } M.$$

- Define $\Delta(N, \Pi) = \sum_i p_i \cdot \Delta(N, M_i)$.

Definition. Call a mixed matching Π popular if $\Delta(N, \Pi) \leq 0\ \forall$ matchings N.

An interesting example

This instance has only one popular matching $S = \{a_1b_1, a_2b_2\}$.



$M = \{a_1b_2, a_2b_1\}$ is unpopular; it is defeated by $N = \{a_0b_2, a_1b_1\}$.

- However the mixed matching $\Pi = \{(S, \frac{1}{2}), (M, \frac{1}{2})\}$ is popular:
  - observe that $\Delta(N, \Pi) = \frac{1}{2} \cdot \Delta(N, S) + \frac{1}{2} \cdot \Delta(N, M) = \frac{1}{2} - \frac{1}{2} = 0$.

Thus a popular mixed matching need not be a probability distribution over popular matchings.

The popular fractional matching polytope

In bipartite graphs: a mixed matching $\equiv$ a fractional matching [Birkhoff-von Neumann theorem].

A fractional matching is a point $\vec{x}$ in the matching polytope $\mathcal{M}_G$, i.e., $\vec{x} \in \mathbb{R}^m_{\geq 0}$ and $\sum_{e \in \delta(u)} x_e \leq 1$ for all $u \in A \cup B$.

$$\text{Let } \mathcal{F}_G = \{\vec{x} \in \mathcal{M}_G : \Delta(N, \vec{x}) \leq 0 \text{ for all matchings } N\}$$

$$\text{where } \Delta(N, \vec{x}) = \sum_{u \in A \cup B} \underbrace{\left(\sum_{v \prec_u N(u)} x_{uv} - \sum_{v \succ_u N(u)} x_{uv} \right)}_{u\text{'s vote for } N(u) \text{ vs } \vec{x}_u}$$

- $\mathcal{F}_G$ is the popular fractional matching polytope of G.
- That is, $\mathcal{F}_G$ is the convex hull of all popular fractional matchings in G.

Towards a compact extended formulation for $\mathcal{F}_G$

For any $\vec{x} \in \mathcal{M}_G$ and $e = ab \in E$:

- Let $\mathrm{wt}_x(e) = (a\text{'s vote for } b \text{ versus } \vec{x}_a) + (b\text{'s vote for } a \text{ versus } \vec{x}_b)$.

$$\text{So } \mathrm{wt}_x(e) = \underbrace{\left(\sum_{b' \prec_a b} x_{ab'} - \sum_{b' \succ_a b} x_{ab'}\right)}_{a\text{'s vote for } b \text{ vs } \vec{x}_a} + \underbrace{\left(\sum_{a' \prec_b a} x_{a'b} - \sum_{a' \succ_b a} x_{a'b}\right)}_{b\text{'s vote for } a \text{ vs } \vec{x}_b}.$$

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- Similarly, $\mathrm{wt}_x(uu) = \underbrace{-\sum_{e \in \delta(u)} x_e}_{u\text{'s vote for itself vs } \vec{x}_u}$ for any $u \in A \cup B$.

Thus $\Delta(N, \vec{x}) = \mathrm{wt}_x(N)$ for any perfect matching N.

(note that N is augmented with self-loops to make it perfect)

- Hence $\vec{x}$ is popular $\iff$ $\mathrm{wt}_x(N) \le 0$ for all perfect matchings N.

LP for max-weight perfect matching

$$\max \sum_{e} \mathsf{wt}_x(e) \cdot y_e$$

$$\begin{aligned} \sum_{e \in \delta(u) \cup \{uu\}} y_e &= 1 \quad \forall u \in A \cup B \\ y_e &\geq 0 \quad \forall e \in E \cup \{\text{self-loops}\}. \end{aligned}$$

$\vec{x}$ is popular $\iff$ opt $= 0$.

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$\vec{x}$ is popular $\iff$ opt $= 0$.

The above LP allows us to test if the fractional matching $\vec{x}$ is popular or not.

- But we want to describe the polytope of all popular fractional matchings.

So we also have to add the constraints $\sum_{e \in \delta(u) \cup \{uu\}} x_e = 1 \ \forall u$ and $x_e \geq 0 \ \forall e$.

- Then the objective function becomes *quadratic* in variables x_e and y_e.

LP for max-weight perfect matching

So let us consider the dual LP for max-weight perfect matching.

Dual LP

$$\min \sum_{u} \alpha_u$$

$$\begin{aligned} \alpha_a + \alpha_b &\geq \mathsf{wt}_x(ab) \quad \forall\ ab \in E \\ \alpha_u &\geq \mathsf{wt}_x(uu) \quad \forall\ u \in A \cup B. \end{aligned}$$

$\vec{x}$ is popular $\iff$ $\exists$ dual feasible $\vec{\alpha}$ with $\sum_u \alpha_u = 0$ (dual certificate for $\vec{x}$).

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Let us add the constraints $\sum_{e \in \delta(u) \cup \{uu\}} x_e = 1 \ \forall u$ and $x_e \geq 0 \ \forall e$ to this LP.

- So this will be an LP in variables x_e's and α_u's.
- The set of optimal solutions $(\vec{x}, \vec{\alpha})$ is an extension of our polytope $\mathcal{F}_G$.

The popular fractional matching polytope $\mathcal{F}_G$

A compact extended formulation

$$\begin{aligned}
\alpha_a + \alpha_b &\geq \mathrm{wt}_x(ab) \quad \forall\ ab \in E \\
\alpha_u &\geq \mathrm{wt}_x(uu) \quad \forall\ u \in A \cup B \\
\sum_{e \in \delta(u) \cup \{uu\}} x_e &= 1\ \forall\ u \in A \cup B \\
x_e &\geq 0\ \forall\ e \in E \cup \{\textit{self-loops}\} \\
\sum_{u \in A \cup B} \alpha_u &= 0.
\end{aligned}$$

We can optimize over the above polytope (call it $\mathcal{F}'_G$) in polynomial time.

Thus we can find a min-cost popular mixed matching in polynomial time.

- A drawback of generalizing to mixed matchings is that the solution has become more complex to describe and difficult to implement.

The polytope $\mathcal{F}'_G$

The LP whose set of optimal solutions is $\mathcal{F}'_G$ has an unusual property:

- It is self-dual.
- So $x_{ab} > 0 \implies \alpha_a + \alpha_b = \mathrm{wt}_x(ab)$.
  (by complementary slackness)

Suppose G admits a perfect stable matching. Let $|A \cup B| = n$.

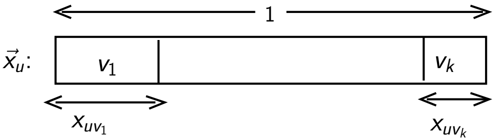
- Then every popular matching in G has a dual certificate in $\{\pm 1\}^n$.
- We show that $\mathcal{F}'_G$ (and thus $\mathcal{F}_G$) is integral in this special case.
  [Huang and K, 2021]

Our method is inspired by the proof of integrality of the formulation of the stable matching polytope [Teo and Sethuraman, 1998].

Integrality of $\mathcal{F}_G$ in the special case

Let $\vec{x} \in \mathcal{F}_G$. Then $(\vec{x}, \vec{\alpha}) \in \mathcal{F}'_G$ for some $\vec{\alpha} \in [-1, 1]^n$.

$\vec{x}_u = (x_{uv_1}, \ldots, x_{uv_k})$ is u's allocation in $\vec{x}$, for every $u \in A \cup B$.

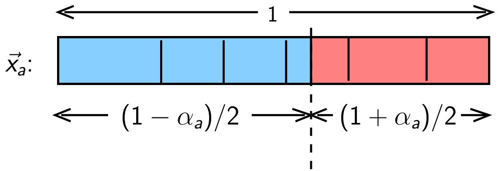


So $v_1, \ldots, v_k$ are the neighbors of u such that $x_{uv_i} > 0$ for $i = 1, \ldots, k$.

- For $a \in A$: arrange the entries of $\vec{x}_a$ in decreasing order of a's preference.
- For $b \in B$: arrange the entries of $\vec{x}_b$ in increasing order of b's preference.
- There is some $\alpha_u \in [-1, 1]$ for each $u \in A \cup B$. How do we interpret α_u?

Reordering the array $\vec{x}_a$

We will use a's α-value as follows.



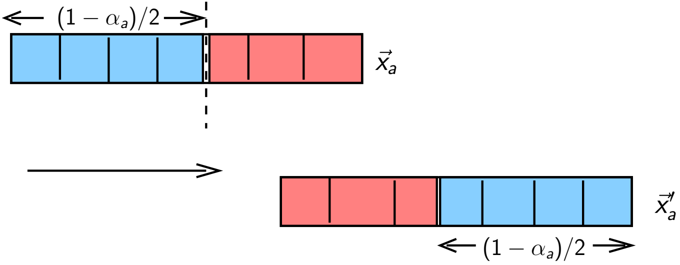
Call the initial $(1-\alpha_a)/2$ fraction of $\vec{x}_a$ the blue sub-array of $\vec{x}_a$.

Call the remaining $(1+\alpha_a)/2$ fraction of $\vec{x}_a$ the red sub-array of $\vec{x}_a$.

Swap the blue and red sub-arrays to form a reordered array $\vec{x}'_a$.

Reordering the array $\vec{x}_a$

Thus $\vec{x}_a \rightsquigarrow \vec{x}'_a$ by this swap.



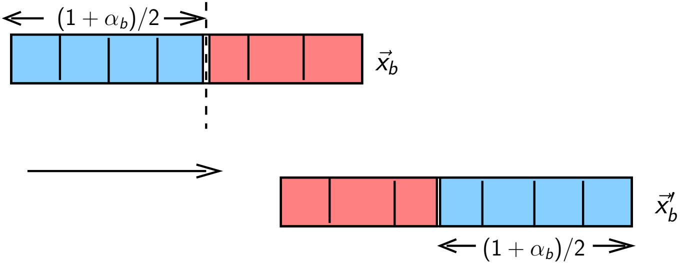
The order within the blue sub-array (similarly, the red sub-array) is preserved.

Reordering the array $\vec{x}_b$

We do an analogous transformation for $\vec{x}_b \rightsquigarrow \vec{x}'_b$ for any $b \in B$.

The initial $(1+\alpha_b)/2$ fraction of $\vec{x}_b$ will be the blue sub-array of $\vec{x}_b$.

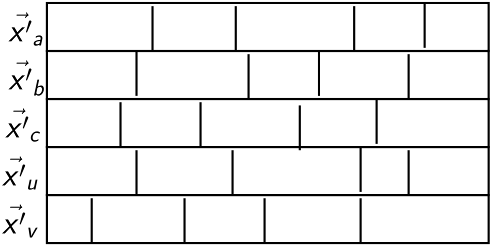
The latter $(1-\alpha_b)/2$ fraction of $\vec{x}_b$ will be the red sub-array of $\vec{x}_b$.



Swap the blue and red sub-arrays to form a reordered array $\vec{x}'_b$.

The table T

Form a table T whose rows are the reordered arrays.

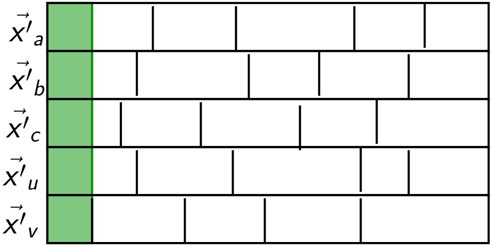


Sweep a vertical line along the table T decomposing it into columns.

$$M_i = \text{pairing defined by the } i\text{th column.}$$

The table T

Form a table T whose rows are the reordered arrays.

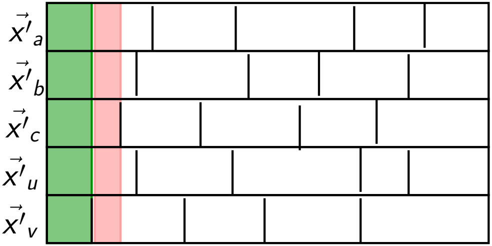


Sweep a vertical line along the table T decomposing it into columns.

M_1 = pairing defined by the first column.

The table T

Form a table T whose rows are the reordered arrays.

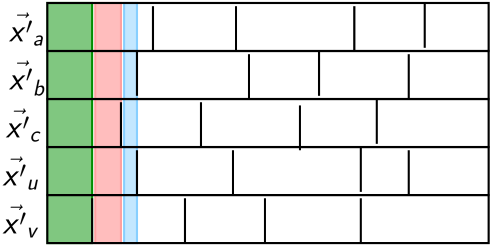


Sweep a vertical line along the table T decomposing it into columns.

M_2 = pairing defined by the second column.

The table T

Form a table T whose rows are the reordered arrays.

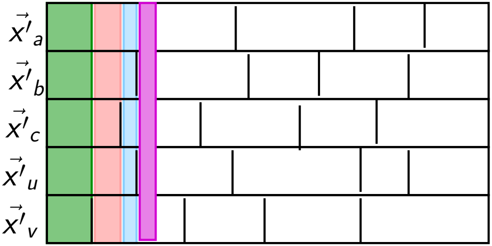


Sweep a vertical line along the table T decomposing it into columns.

M_3 = pairing defined by the third column.

The table T

Form a table T whose rows are the reordered arrays.



Sweep a vertical line along the table T decomposing it into columns.

M_4 = pairing defined by the fourth column.

Popularity of M_i

Thus $\vec{x} = \sum_i p_i \cdot M_i$, where p_i is the width of M_i's column.

- Self-duality of our LP $\Rightarrow$ each M_i is a matching in G.

We show a dual certificate $\vec{\alpha_i} \in \{\pm 1\}^n$ for each M_i.

- M_i corresponds to a red cell in $\vec{x}'_a$ (where $a \in A$) $\Rightarrow \alpha_i(a) = 1$;
- M_i corresponds to a red cell in $\vec{x}'_b$ (where $b \in B$) $\Rightarrow \alpha_i(b) = -1$;
- M_i corresponds to a blue cell in $\vec{x}'_a$ (where $a \in A$) $\Rightarrow \alpha_i(a) = -1$.
- M_i corresponds to a blue cell in $\vec{x}'_b$ (where $b \in B$) $\Rightarrow \alpha_i(b) = 1$.

This vector $\vec{\alpha_i}$ will be a feasible solution to the dual LP.

- Moreover, $\sum_{u \in A \cup B} \alpha_i(u) = 0$.
- Thus $\vec{\alpha_i}$ is a dual certificate for M_i.

Integrality of our formulation

Thus we have $(\vec{x}, \vec{\alpha}) = \sum_i p_i \cdot (M_i, \vec{\alpha_i})$ where:

- $\sum_i p_i = 1$ and $p_i \geq 0 \, \forall \; i$;
- each M_i is a matching;
- $\vec{\alpha}_i \in \{\pm 1\}^n$ is M_i's dual certificate (so M_i is popular).

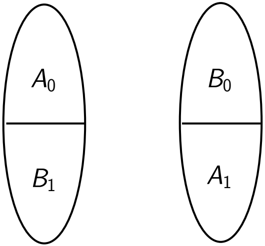
Hence $\mathcal{F}'_G$ (and thus $\mathcal{F}_G$) is integral.

- This is for the special case when G admits a perfect stable matching.

So we know how to formulate the popular matching polytope of G in this special case.

Half-integrality of the polytope $\mathcal{F}_G$ in the general case

Corresponding to $G = (A \cup B, E)$, let us build the following graph H:



Every vertex $u \in A \cup B$ has two copies u_0 and u_1 in H.

H is made up of 2 copies of G along with the "self-loop" edges $u_0 u_1$ $\forall u$.

- H admits a perfect stable matching.
- So $\mathcal{F}_H$ is integral.

Half-integrality of the polytope $\mathcal{F}_G$ in the general case

We can define natural maps $f : \mathcal{F}_G \to \mathcal{F}_H$ and $h : \mathcal{F}_H \to \mathcal{F}_G$:

- f will copy e's x-value on e_0 and e_1: $x_{e_0} = x_{e_1} = x_e$;
- h will be a "halving map": $x_e = (x_{e_0} + x_{e_1})/2$.
- so $h \circ f(\vec{x}) = \vec{x}$ for any $\vec{x} \in \mathcal{P}_G$.

Integrality of $\mathcal{F}_H \implies f(\vec{x}) = \{(M_i, p_i) : i = 1, \ldots, k\}$ for popular matchings $M_1, \ldots, M_k$ in H.

- So $\vec{x} = h \circ f(\vec{x}) = \{(h(M_i), p_i) : i = 1, \ldots, k\}$.

Half-integrality of the polytope $\mathcal{F}_G$ in the general case

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- So $\vec{x} = h \circ f(\vec{x}) = \{(h(M_i), p_i) : i = 1, \ldots, k\}$.

Each $h(M_i)$ will be a popular half-integral matching in G.

- Thus $\vec{x}$ is a convex combination of popular half-integral matchings.
- So the popular fractional matching polytope $\mathcal{F}_G$ is half-integral.
- Hence there is always a min-cost popular mixed matching $\Pi = \{(M_0, \frac{1}{2}), (M_1, \frac{1}{2})\}$.

The popular matching polytope

Let $\mathcal{P}_G$ be the popular matching polytope of G.

- The extension complexity of $\mathcal{P}_G$ is $2^{\Omega\left(\frac{m}{\log m}\right)}$, where $|E| = m$. [Faenza and K, 2022]

Can we relax *popular* to "approximately popular" for the sake of tractability?

- M is popular $\implies$ no matching wins more votes than M.

DEFINITION. Call a matching M quasi-popular if no matching wins more than twice as many votes as M.

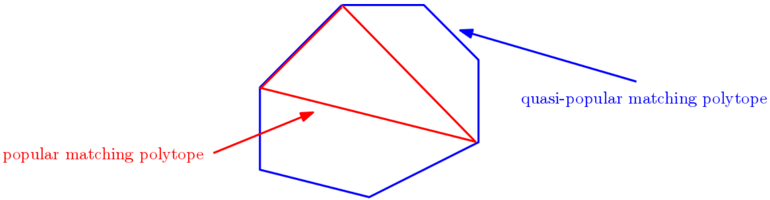
- Can a min-cost quasi-popular matching be efficiently computed?
- No, we show it is NP-hard to compute a min-cost quasi-popular matching.

Quasi-popular matchings

The min-cost popular / quasi-popular matching problems are NP-hard to approximate up to any factor.

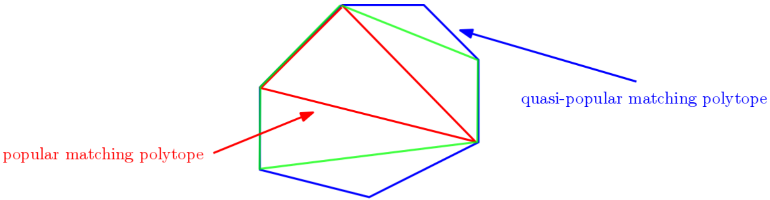
- A Bicriteria Approximation. Can we efficiently find a quasi-popular matching of cost at most that of a min-cost popular matching?
- Interestingly, the answer is *yes*.

Our technique



- The polytopes $\mathcal{P}_G$ and $\mathcal{Q}_G$ have near-exponential extension complexity.

Our technique



- The polytopes $\mathcal{P}_G$ and $\mathcal{Q}_G$ have near-exponential extension complexity.
- We show an integral polytope $\mathcal{C}$ sandwiched between $\mathcal{P}_G$ and $\mathcal{Q}_G$ such that $\mathcal{C}$ has a compact extended formulation.
- Optimizing over $\mathcal{C}$ leads to the efficient bicriteria algorithm.

The popular matching polytope in a special case

Special case:

- G has a perfect stable matching $\implies \mathcal{F}_G$ is integral.

From general case to special case:

- Assume for simplicity that $|A| = |B|$.
- **Our idea:** augment G with some new edges so that the resulting graph G^* has a perfect stable matching.

The popular matching polytope in a special case

Special case:

- G has a perfect stable matching $\implies \mathcal{F}_G$ is integral.

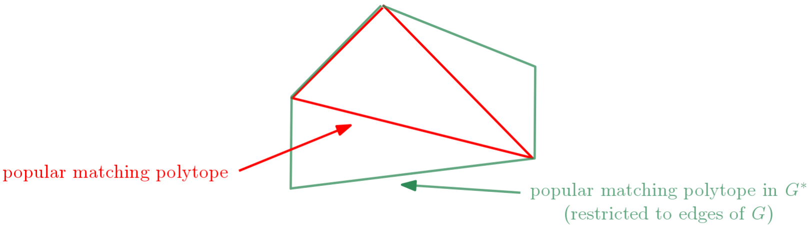
From general case to special case:

- Assume for simplicity that $|A| = |B|$.
- **Our idea:** augment G with some new edges so that the resulting graph G^* has a perfect stable matching.
- Pair up unstable vertices appropriately; add a new edge between each pair.

So the popular matching polytope of G^* has a compact extended formulation.
(by [Huang and K, 2021])

The sandwiched integral polytope

Every popular matching in G can be extended to a perfect popular matching in G^* using these new edges.



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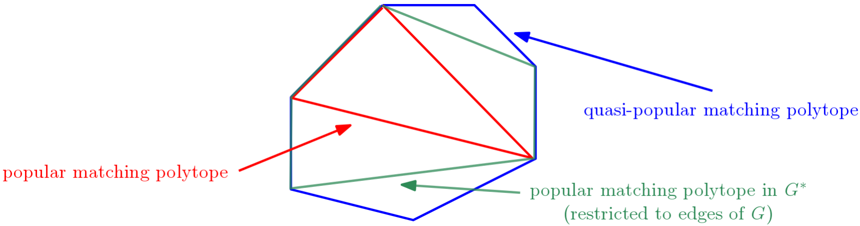
Every popular matching in G can be extended to a perfect popular matching in G^* using these new edges.

Every popular matching in G^*, when restricted to edges of G, is quasi-popular in G.



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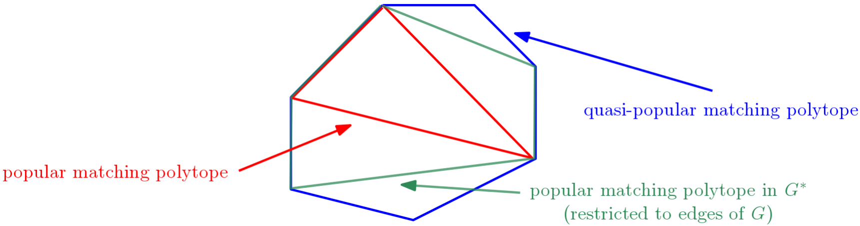

G^* has a perfect stable matching, so $\mathcal{F}_{G^*}$ = popular matching polytope of G^*.

The compact extended formulation of $\mathcal{F}_{G^*}$ is an extension of an integral polytope $\mathcal{C}$ where: $\mathcal{P}_G \subseteq \mathcal{C} \subseteq \mathcal{Q}_G$.

Another relaxation of popularity

Suppose N is a "very unpopular" matching.

- Let us not give N the power to block other matchings.
- That is, we waive the constraint $\Delta(N, \vec{x}) \leq 0$ in the popular matching polytope formulation to get a more relaxed formulation.

We seek a relaxation that admits a compact formulation.

- Then we can efficiently optimize over this polytope.

How do we define "very unpopular" matchings?

Another relaxation of popularity

Call a matching S supporting if $\exists$ a popular mixed matching Π whose support contains S.

- So $\exists$ popular $\Pi = \{(M_0, p_0), \ldots, (M_k, p_k)\}$ such that $S = M_i$ for some i.

A non-supporting matching cannot form a popular mixture even with the help of other matchings.

- A matching that is not supporting will be considered "very unpopular".

Call a matching M fairly popular if $\Delta(S, M) \leq 0$ $\forall$ supporting matchings S.

- Though M need not be popular, any matching that defeats M is *uninteresting* wrt popularity.

Fairly popular matchings

Can a min-cost fairly popular matching be computed in polynomial time?

- Interestingly, the answer is *yes*.

Fairly popular matchings

Can a min-cost fairly popular matching be computed in polynomial time?

- Interestingly, the answer is *yes*.

The following statements are equivalent.

- S is a supporting matching.
- No popular mixed matching defeats S.
- S matches all stable vertices and $S \subseteq E_p$ where E_p is the set of popular fractional edges in G.

The set E_p is the set of popular edges in the graph H (this is two copies of G).

Fairly popular matchings

This simple characterization of supporting matchings allows us to show:

- M is fairly popular $\iff$ M can be realized as a stable matching in a certain colorful multigraph.
- The min-cost stable matching algorithm in this multigraph finds a min-cost fairly popular matching in G in polynomial time.
- The fairly popular matching polytope admits a compact extended formulation as the stable matching polytope of this colorful multigraph.

A further relaxation

Call a matching M pseudo-popular if $\Delta(P, M) = 0$ $\forall$ popular matchings P.

- So M is undefeated by all popular matchings.
- What is the complexity of deciding if a given matching M is pseudo-popular?
- This is a coNP-hard problem.

A further relaxation

Call a matching M pseudo-popular if $\Delta(P, M) = 0$ $\forall$ popular matchings P.

- So M is undefeated by all popular matchings.
- What is the complexity of deciding if a given matching M is pseudo-popular?
- This is a coNP-hard problem.

By contrast, it is easy to decide if a given matching M is undefeated by all popular mixed matchings.

- Such an M is a supporting matching.

  (by our characterization of supporting matchings)
- So M has to match all stable vertices and $M \subseteq E_p$.

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Thank you! Any questions?